# Muckenhoupt-Wheeden conjectures for sparse operators

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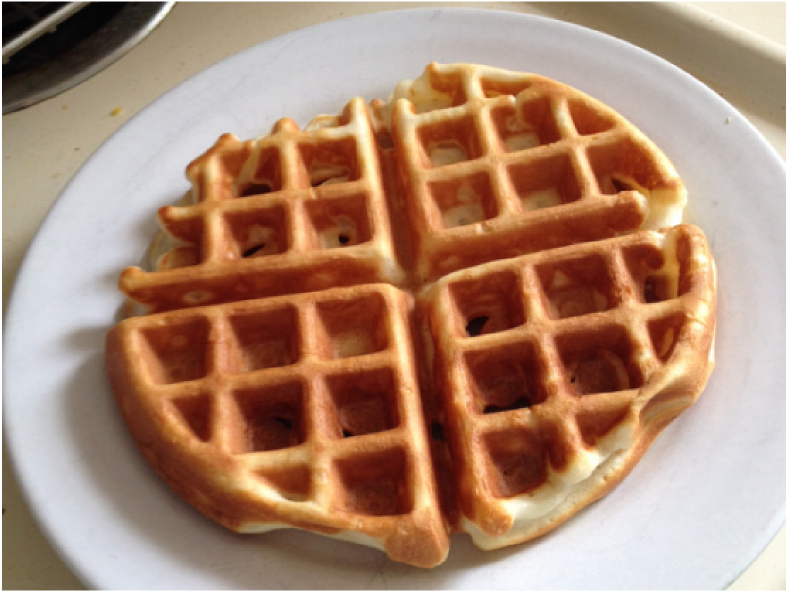
University of Alabama, Tuscaloosa

2017 – University of New Haven – Seminar Talk

Based on a joint work with my advior, Prof. Kabe Moen.

# Dyadic grid



## $L^p$-spaces

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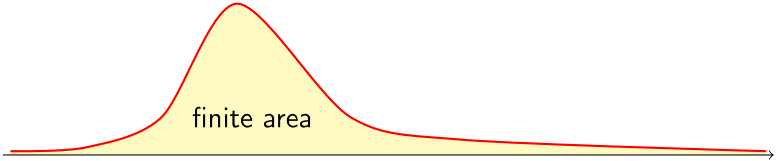

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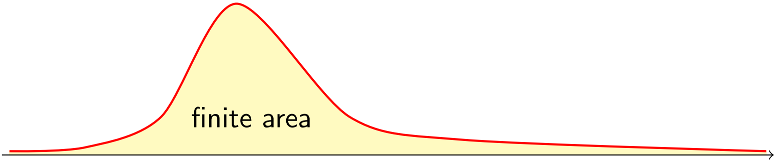


- If $p \geqslant 1$, $L^p(\mathbb{R}^n)$ is normed with

$$\|f\|_{L^p(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |f(y)|^p \, dy \right)^{\frac{1}{p}}$$

# Weak $L^p$-spaces, a.k.a. $L^{p,\infty}$

- $f \in L^{p,\infty}(\mathbb{R}^n)$ iff

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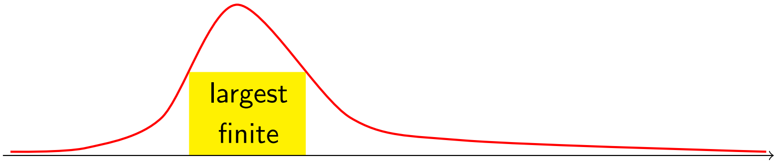

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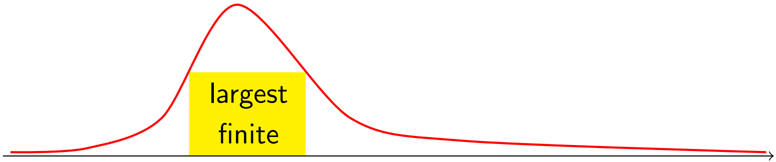


- Quasi-norm:

$$\|f\|_{L^{p,\infty}(\mathbb{R}^n)} = \sup_{\lambda>0} \lambda \cdot \left|\{x \in \mathbb{R}^n : |f(x)| > \lambda\}\right|^{\frac{1}{p}}$$

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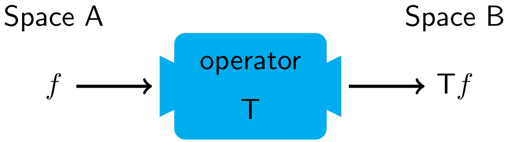
# Operator



# Operator


- If A and B are normed, T is bounded from A to B $(T: A \longrightarrow B)$ iff

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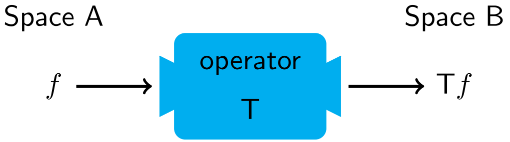


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- Operator norm:

$$\|T\|_{A \to B} = \inf\{c : \|Tf\|_B \leqslant c \cdot \|f\|_A\}$$

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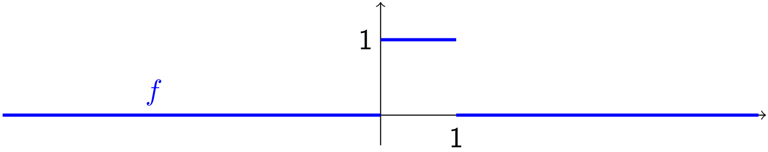

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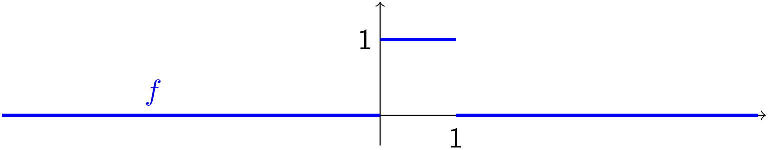


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$$\frac{1}{|Q|} \int_Q \mathbb{1}_{[0,1]}(y)\, dy = \frac{|Q \cap [0,1]|}{|Q|} \leqslant 1$$

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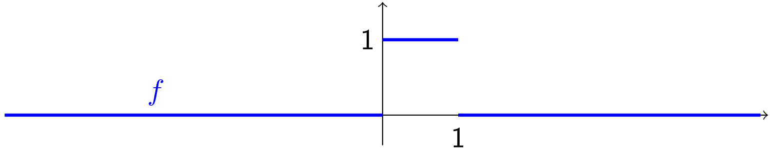


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- When $x \in [0,1]$, pick $Q = [0,1]$ to see $\mathsf{M}f(x) = 1$.

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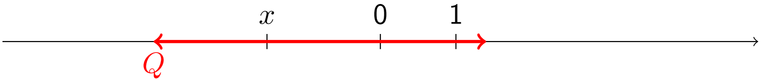

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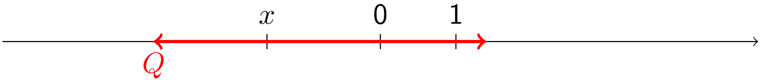


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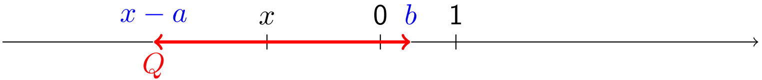

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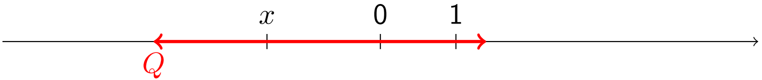


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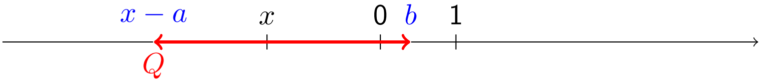


  $\mathrm{M}f(x) = \frac{1}{1-x}$

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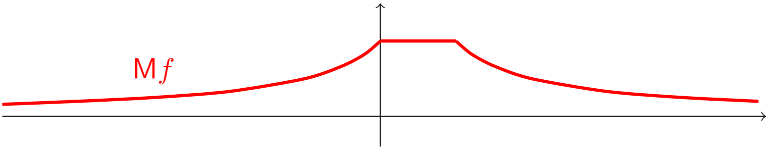
- When $x > 1$, $\mathsf{M}f(x) = \frac{1}{x}$
- All combined:

$$\mathsf{M}f(x) = \begin{cases} \frac{1}{1-x} & \text{if} \quad x < 0 \\ 1 & \text{if} \quad 0 \leqslant x \leqslant 1 \\ \frac{1}{x} & \text{if} \quad x > 1 \end{cases}$$



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- Un-weighted:

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- Example:

$$w \in \mathsf{A}_2 \iff \sup_Q \left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q w^{-1}\right) < \infty$$

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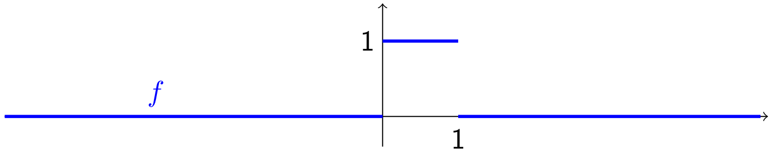

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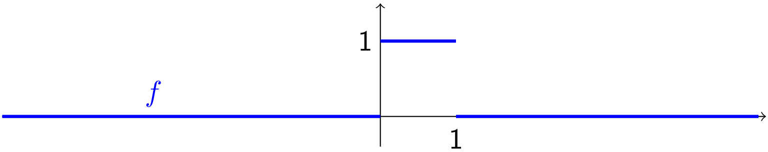


- Observe:

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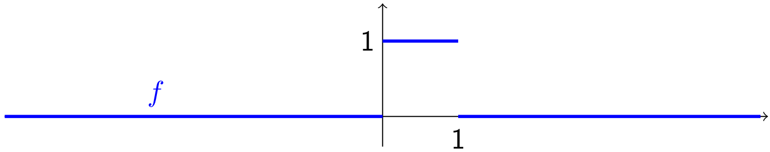


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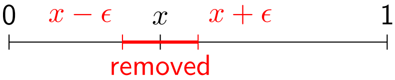
- $\mathsf{H}f(0) = -\infty$, and $\mathsf{H}f(1) = \infty$.

# Hilbert transform

- When $x \in (0,1)$




$$\mathsf{H}f(x) = \lim_{\epsilon\to 0}\left[\int_0^{x-\epsilon}\frac{1}{x-y}\,dy + \int_{x+\epsilon}^{1}\frac{1}{x-y}\,dy\right] = \ln\left|\frac{x}{x-1}\right|$$

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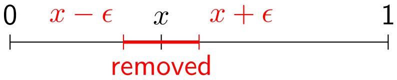


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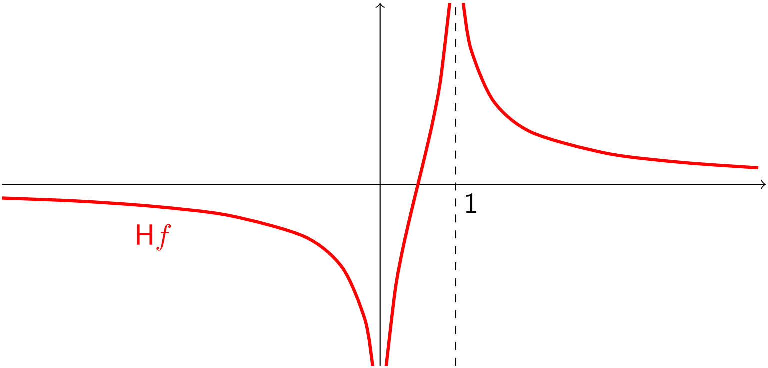

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# Calderón-Zygmund singular integral operator (CZO)

$$\mathsf{T}f(x) = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} \mathsf{K}(x,y) f(y)\, dy$$

where

$$|\mathsf{K}(x,y)| \lesssim \frac{1}{|x-y|^n}$$

and there exists $\delta > 0$

$$|\mathsf{K}(x,y) - K(x,z)| + |\mathsf{K}(y,x) - \mathsf{K}(z,x)| \lesssim \frac{|y-z|^\delta}{|x-y|^{n+\delta}} \text{ for } |x-y| > 2|y-z|$$

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# Two-weight $A_p$ condition

- Example: $(u, v) \in A_2$ iff

$$\sup_Q \left( \frac{1}{|Q|} \int_Q u \right) \left( \frac{1}{|Q|} \int_Q v^{-1} \right) < \infty$$

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- Two-weight $A_p$ is NOT sufficient for anything else.

# The raise of the conjectures

$$(u, v) \in \mathsf{A}_2$$

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$$\begin{array}{ll} \mathsf{M} : L^2(v) \to L^2(u) & \\ & \searrow\!\!\!\!\searrow \\ & (u,v) \in \mathsf{A}_2 \end{array}$$

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$$
\begin{array}{l}
\mathsf{M}: L^2(v) \to L^2(u) \\
\qquad\qquad\qquad\qquad \Downarrow \\
\qquad\qquad\qquad\qquad (u,v) \in \mathsf{A}_2 \\
\qquad\qquad\qquad\qquad \Uparrow \\
\mathsf{M}: L^2(u^{-1}) \to L^2(v^{-1})
\end{array}
$$

# The raise of the conjectures

$$\begin{tikzcd}
\mathsf{M} : L^2(v) \to L^2(u) \arrow[dr, Rightarrow] & & \mathsf{T} : L^2(v) \to L^2(u) \\
& (u,v) \in \mathsf{A}_2 \arrow[ur, Rightarrow, "?"] & \\
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\end{tikzcd}$$

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- Muckenhoupt-Wheeden conjecture for CZO:

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- 1-weight endpoint conjecture:

$$w\left(\{x \in \mathbb{R}^n : |\mathsf{T}f(x)| > \lambda\}\right) \leqslant \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)|\, \mathsf{M}w(x)\, dx$$

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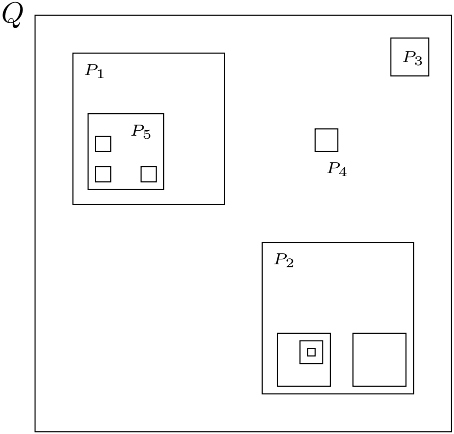
- 2013, Criado and Soria: higher dimensions.

- Convention: $f \in L^p(v)$ means supp$(f) \subset$ supp$(v)$.

# Sparse family of cubes

$\mathcal{S}$ is sparse iff:

$$\left| \bigcup_{\substack{P \in \mathcal{S} \\ P \subsetneq Q}} P \right| \leq \frac{1}{2} |Q|, \qquad Q \in \mathcal{S}.$$



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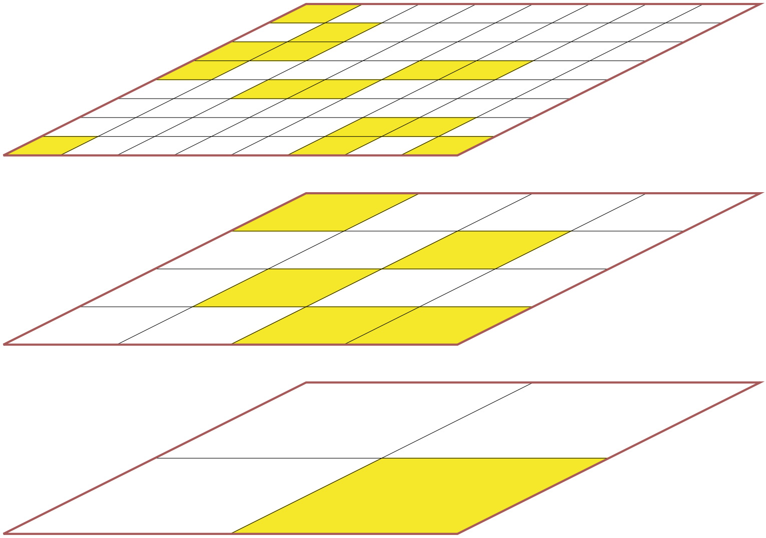

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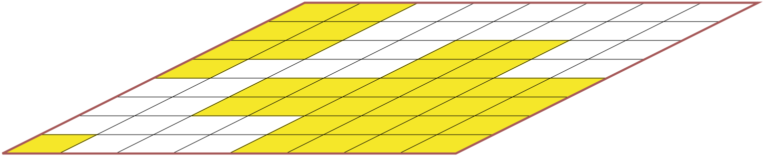

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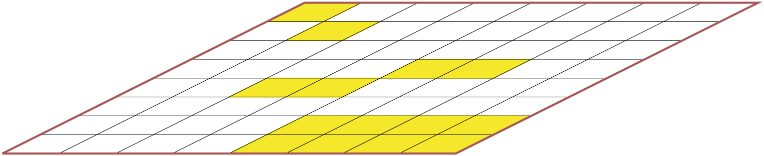

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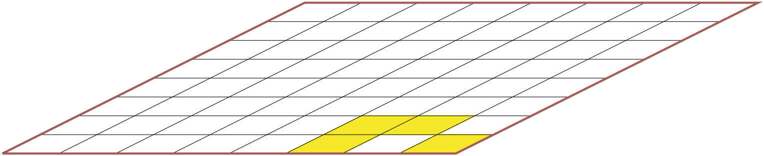

# Spares operator

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$$|\mathsf{T} f| \lesssim \sum_{i=1}^{N} \mathsf{T}_{\mathcal{S}_i} |f|$$

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- Non-constructive, which $\mathcal{S}_i$ **?**

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A constructive disproof:

- Building-block $w_k$
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- Visualized with $n = 2$ and $k = 3$

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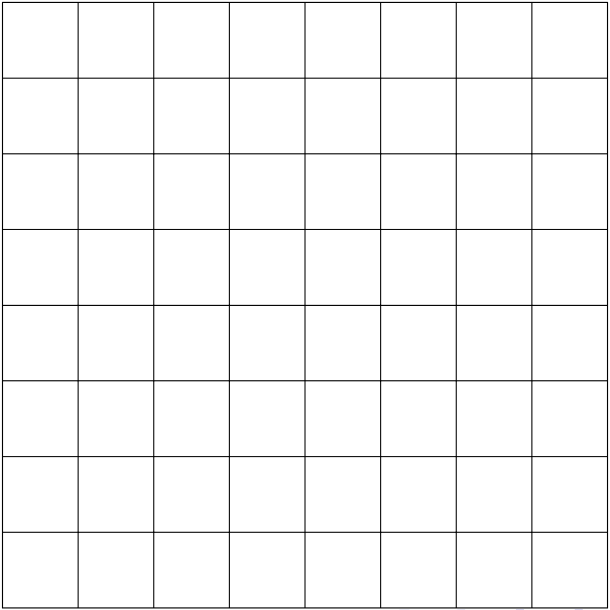

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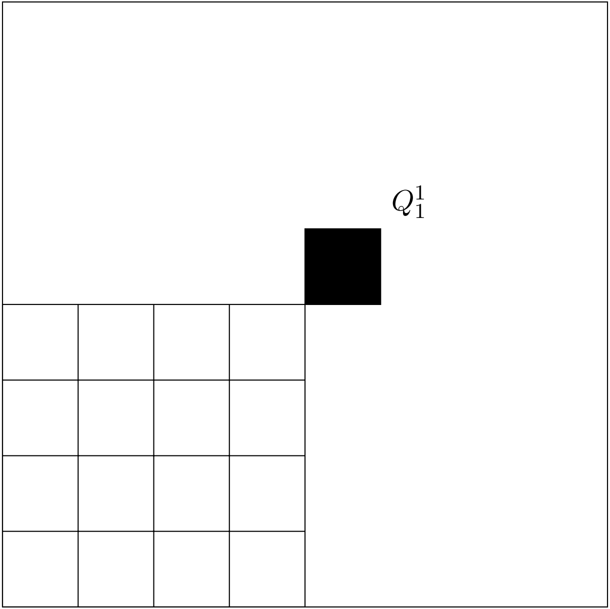

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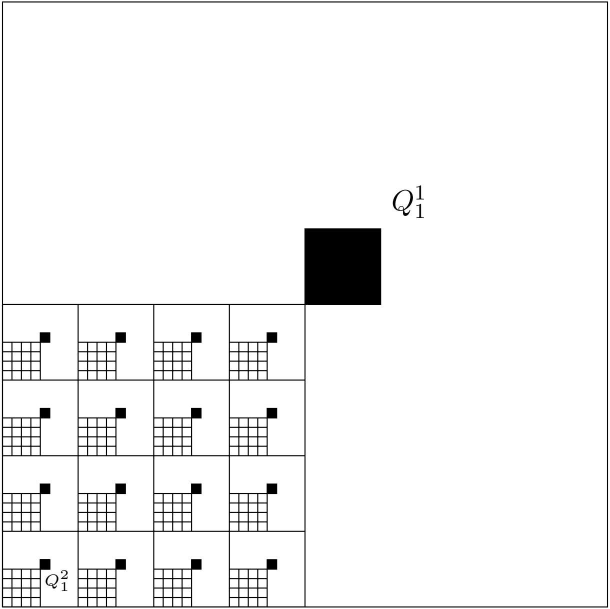

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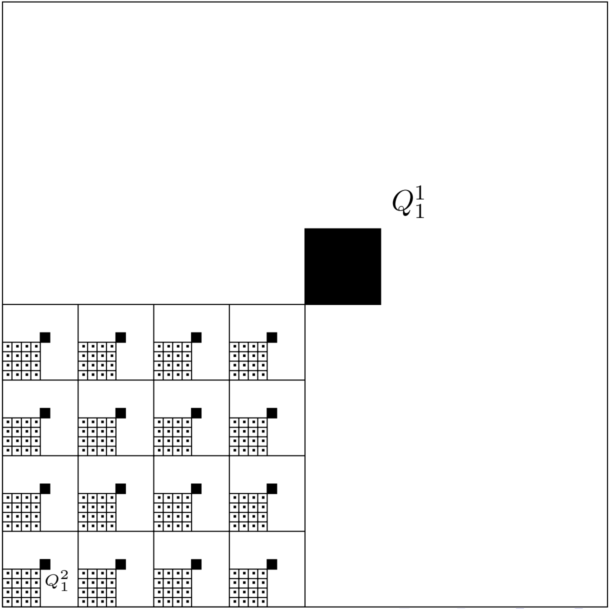

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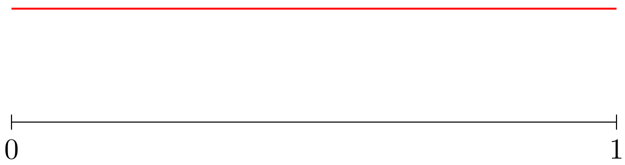

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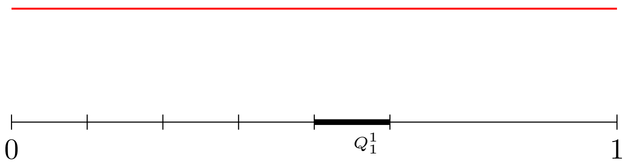

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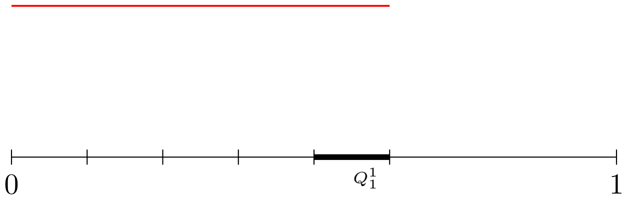

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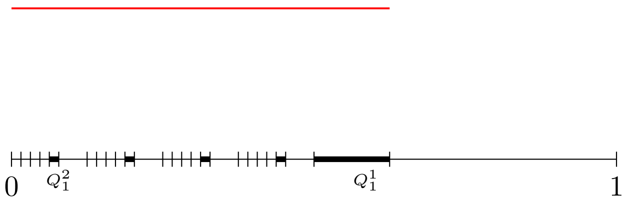

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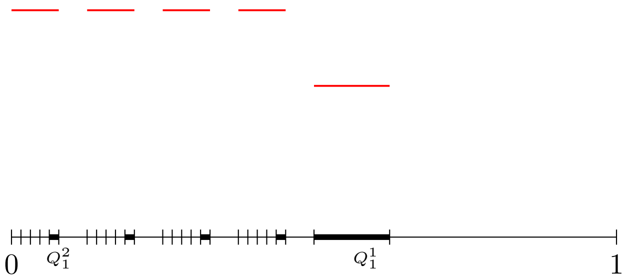

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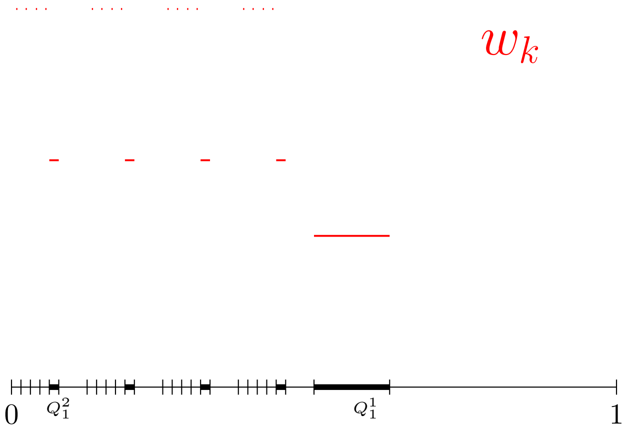

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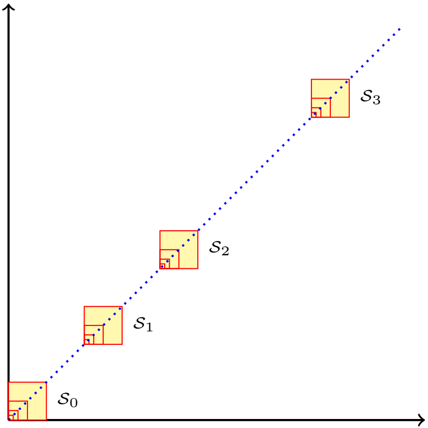
- $\mathrm{M}(w_k)(x) \leqslant 9^n\, w_k(x)$ for all $x \in \cup_{m=1}^{\infty} \Omega_m$.

# The Sparse family $\mathcal{S}$

- $\mathcal{S} = \bigcup_{k=0}^{\infty} \mathcal{S}_k$ where: $\quad \mathcal{S}_0 = \{[0, 2^{-N})^n : \ N = 0, 1, 2, ...\}$, and

$$\mathcal{S}_k = \{[2^k, 2^k + 2^{-N})^n : \ N = 0, 1, 2, ...\}.$$



# The Sparse family $\mathcal{S}$

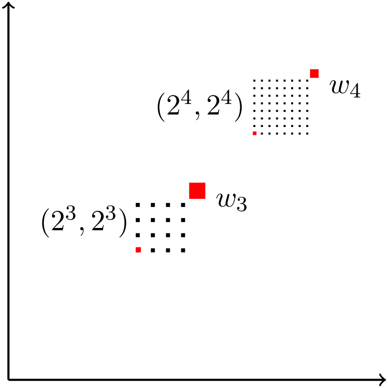
**Lemma 1**

$\mathsf{T}_{\mathcal{S}}(w_k)(x) \geqslant \frac{k}{2(2^n-1)} w_k(x)$ *for* $k \geqslant 3$ *and* $x \in \bigcup_{m=3}^{\infty} Q_1^m$.

## Weights $(u, v)$

- $\Gamma(k) = \bigcup_{m=3}^{\infty} Q_1^m(k)$, $A_k = \frac{1}{w_k(\Gamma(k))}$, define

$$v(x) = \sum_{k=3}^{\infty} A_k\, w_k(x - 2^{\vec{k}})\; \mathbb{1}_{\Gamma(k)}(x - 2^{\vec{k}})$$



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- With $(u, v)$:

$$\mathsf{M} : L^p(v) \to L^p(u)$$

$$\mathsf{M} : L^{p'}(u^{1-p'}) \to L^{p'}(v^{1-p'})$$

but

$$\mathsf{T}_{\mathcal{S}} : L^p(v) \not\to L^p(u)$$

# Disprove the conjecture

- T.F.A.E.

$$\|T_{\mathcal{S}}f\|_{L^p(u)} \lesssim \|f\|_{L^p(v)}$$
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Then

$$\|f\|_{L^{p'}(v^{1-p'})}^{p'} = \sum_{k=3}^{\infty} \frac{1}{k^{\epsilon p'}} < \infty$$

while

$$\|T_{\mathcal{S}} f\|_{L^{p'}(u^{1-p'})}^{p'} \geqslant \sum_{k=3}^{\infty} k^{(1-\epsilon)p'} = \infty$$

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Assuming that $x \in Q_1^m$.

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\begin{aligned}
T_{\mathcal{S}}(w_k)(x) &= \sum_{Q \in S} \frac{1}{|Q|} \int_Q w_k(y)\,dy \;\; \mathbb{1}_Q(x) \\
&= \sum_{Q \in S_0} \frac{1}{|Q|} \int_Q w_k(y)\,dy \;\; \mathbb{1}_Q(x) \\
&= \sum_{N=0}^{\infty} 2^{Nn} \int_{[0,2^{-N})^n} w_k(y)\,dy \;\; \mathbb{1}_{[0,2^{-N})^n}(x) \\
&= \sum_{N=0}^{k(m-1)} 2^{Nn} \int_{[0,2^{-N})^n} w_k(y)\,dy \\
&= 1 + \sum_{i=1}^{m-1} \sum_{j=1}^{k} 2^{(j+k(i-1))n}\, w_k\Big([0, 2^{-j-k(i-1)})^n\Big)
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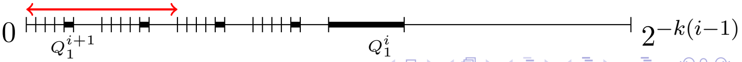

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$$\begin{aligned}
T_{\mathcal{S}}(w_k)(x) &= 1 + \sum_{i=1}^{m-1}\sum_{j=1}^{k}\left(\frac{2^{kn}}{2^{(k-1)n}+1}\right)^i \\
&= 1 + k\cdot\frac{2^{(k-1)n}+1}{(2^n-1)2^{(k-1)n}-1}\cdot\left[\left(\frac{2^{kn}}{2^{(k-1)n}+1}\right)^m - \frac{2^{kn}}{2^{(k-1)n}+1}\right] \\
&= 1 + \frac{k}{2}\cdot\frac{2^{(k-1)n}+1}{(2^n-1)2^{(k-1)n}-1}\cdot\left(\frac{2^{kn}}{2^{(k-1)n}+1}\right)^m \\
&\quad + \frac{k}{2}\cdot\frac{2^{(k-1)n}+1}{(2^n-1)2^{(k-1)n}-1}\cdot\left[\left(\frac{2^{kn}}{2^{(k-1)n}+1}\right)^m - \frac{2^{kn+1}}{2^{(k-1)n}+1}\right] \\
&\geqslant \frac{k}{2(2^n-1)}\, w_k(x).
\end{aligned}$$

# Proof of $\|T_{\mathcal{S}} f\|^{p'}_{L^{p'}(u^{1-p'})} = \infty$

$$
\begin{aligned}
\|T_{\mathcal{S}} f\|^{p'}_{L^{p'}(u^{1-p'})} &= \int_{\mathbb{R}^n} T_{\mathcal{S}} f(x)^{p'} \frac{v(x)}{Mv(x)^{p'}}\, dx \\
&\geqslant C \int_{\mathbb{R}^n} T_{\mathcal{S}} f(x)^{p'}\, v(x)^{1-p'} dx \\
&= C \sum_{k=3}^{\infty} A_k^{1-p'} \int_{2^{\vec{k}}+\Gamma(k)} T_{\mathcal{S}} f(x)^{p'} w_k(x - 2^{\vec{k}})^{1-p'}\, dx \\
&= C \sum_{k=3}^{\infty} A_k^{1-p'} \int_{\Gamma(k)} T_{\mathcal{S}} f(x + 2^{\vec{k}})^{p'} w_k(x)^{1-p'}\, dx \\
&\geqslant C \sum_{k=3}^{\infty} A_k k^{(1-\epsilon)p'} w_k(\Gamma(k)) = \sum_{k=3}^{\infty} k^{(1-\epsilon)p'} = \infty.
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T_{\mathcal{S}}f(x+2^{\vec{k}}) &= \sum_{Q\in S_k} \frac{1}{|Q|}\int_Q f(y)\,dy\ \mathbb{1}_Q(x+2^{\vec{k}}) \\
&= \sum_{P\in S_0} \frac{1}{|P|}\int_{2^{\vec{k}}+P} f(y)\,dy\ \mathbb{1}_P(x) \\
&= \sum_{P\in S_0} \frac{1}{|P|}\int_{2^{\vec{k}}+P} \frac{1}{k^{\epsilon}}\,A_k\,w_k(y-2^{\vec{k}})\,dy\ \mathbb{1}_P(x) \\
&= \frac{1}{k^{\epsilon}}\,A_k \sum_{P\in S_0} \frac{1}{|P|}\int_P w_k(z)\,dz\ \mathbb{1}_P(x) \\
&= \frac{1}{k^{\epsilon}}\,A_k\,T_{\mathcal{S}}(w_k)(x).
\end{aligned}
$$

Thank you for your attention!