Investigation of holomorphic functions on the bidisk via operator theory

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February 21, 2017
University of New Haven
Mathematics Seminar

Schur functions

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A function $\varphi \in \mathcal{S}_1$ is well-behaved on the interior of the disk $\mathbb{D}$. What about at the edge?

Studying Schur functions

Question

How do Schur functions behave on the boundary of the unit circle $\mathbb{T}$? When do derivatives exist at boundary points $\tau \in \mathbb{T}$? Is there any structure to the derivatives at boundary points?

These questions are typical of functional analysis, which studies classes of functions.

A useful difference quotient

We are interested in the existence of limits and derivatives at boundary points. So we define a difference quotient that examines when the function stays under control near the boundary:

Definition

The **Julia quotient** for a function $\varphi \in \mathcal{S}_1$ is the ratio

$$J_\varphi(\lambda) = \frac{1 - |\varphi(\lambda)|}{1 - |\lambda|}.$$

Carapoints I

Definition

Let $\varphi \in \mathcal{S}_1$. A point $\tau \in \mathbb{T}$ is a **carapoint** for φ if there exists a sequence $\{\lambda_n\} \subset \mathbb{D}$ tending to τ such that

$$J_\varphi(\lambda_n) = \frac{1 - |\varphi(\lambda_n)|}{1 - |\lambda_n|}$$

is bounded.

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When looking at boundary points, we need to avoid introducing additional boundary behavior into the analysis. Thus, we will use **non-tangential limits**:

- a set S approaches $\tau \in \mathbb{T}$ nontangentially if S is contained in a wedge with a point at τ.
- A sequence $\{\lambda_n\} \subset S$ tends to τ nontangentially if $\lim_{n\to\infty} \lambda_n = \tau$ and $\{\lambda_n\}$ is a nontangential set at τ. We write $\lambda \stackrel{\mathrm{nt}}{\to} \tau$ to indicate a non-tangential limit.

Julia-Carathéodory Theorem

Theorem (Julia, Carathéodory)

Let φ a nonconstant Schur function, i.e. φ is holomorphic and $\varphi : \mathbb{D} \to \mathbb{D}$. Let τ be a point in $\mathbb{T}$. The following are equivalent:

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C *there exists $\varphi(\tau) \in \mathbb{T}$ such that $\varphi(\tau) = \lim_{\lambda \stackrel{\mathrm{nt}}{\to} \tau} \varphi(\lambda)$,*

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A τ *is a carapoint for* φ*;*

B *for every sequence* $\{\lambda_n\} \stackrel{\mathrm{nt}}{\to} \tau$, $J_\varphi(\lambda_n)$ *is bounded;*

C *there exists* $\varphi(\tau) \in \mathbb{T}$ *such that* $\varphi(\tau) = \lim_{\lambda \stackrel{\mathrm{nt}}{\to} \tau} \varphi(\lambda)$*, and furthermore,* φ *is nontangentially differentiable at* τ*, i.e. there exists* $\varphi'(\tau) \in \mathbb{T}$ *such that, as* $\lambda \stackrel{\mathrm{nt}}{\to} \tau$,

$$\varphi(\lambda) = \varphi(\tau) + \varphi'(\tau)(\lambda - \tau) + o(|\lambda - \tau|).$$

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C *φ is nontangentially differentiable at τ, i.e. there exists $\varphi(\tau), \varphi'(\tau) \in \mathbb{T}$ such that, as $\lambda \stackrel{\mathrm{nt}}{\to} \tau$,*

$$\varphi(\lambda) \cong \varphi(\tau) + \varphi'(\tau)(\lambda - \tau) \qquad .$$

A comment and a question

First, a comment on what the J-C Theorem says: as long as $\varphi(\lambda)$ doesn't run away too quickly towards the boundary as the input λ approaches τ, φ is nice near τ in the sense that it has a linear approximation on nontangential sets.

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Question

Is this theorem true in two variables? What would it have to say?

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If a two variable Schur function φ is controlled by a growth condition near a boundary point τ, then φ is nicely behaved on nontangential sets near τ.

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Let $\mathbb{D}^2 = \mathbb{D} \times \mathbb{D}$ be the complex unit bidisk. We write $\lambda = (\lambda^1, \lambda^2)$ for the components, with $\left|\lambda^i\right| < 1$.

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The distinguished boundary of $\mathbb{D}^2$ is $\mathbb{T}^2 = \mathbb{T} \times \mathbb{T}$, the unit 2-torus. We write $\tau = (\tau^1, \tau^2)$ for the components with $\left|\tau^i\right| = 1$
($\mathbb{T}^2$ is where the interesting function theory occurs.)

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The *Schur class in two variables* $\mathcal{S}_2$ is the family of holomorphic functions from $\mathbb{D}^2$ to $\mathbb{D}^-$.

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The **Julia quotient** for a function $\varphi \in \mathcal{S}_2$ is the ratio

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Carapoints II

Definition

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Conjecture

If $\varphi \in \mathcal{S}_2$ has a carapoint at τ, then φ has a linear approximation on nontangential sets near τ.

A speculation on the bidisk

Speculation

Let $\varphi \in \mathcal{S}_2$*. If* $\tau \in \mathbb{T}^2$ *is a carapoint for* φ*, then* φ *is nontangentially differentiable at* τ*, i.e. there exist a nontangential limit* $\varphi(\tau) \in \mathbb{T}$ *and a gradient* $\nabla\varphi(\tau)$ *such that*

$$\varphi(\lambda) = \varphi(\tau) + \nabla\varphi(\tau) \cdot (\lambda - \tau) + o(\|\lambda - \tau\|)$$

as $\lambda \stackrel{\mathrm{nt}}{\to} \tau$*.*

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On the other hand,

- In two variables, polynomials **rarely** factor into linear terms. (even worse in more than two variables)
- In two variables, the zeroes of polynomials are **never** isolated.

A simple rational function

Example

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φ has a carapoint at χ.
φ has a nontangential limit $\varphi(\chi) = 1$.
BUT φ does NOT have a linear approximation near χ.

Directional derivatives

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$$D_{-\delta}\varphi(\chi) = -\frac{2\delta_1\delta_2}{\delta_1 + \delta_2}.$$

Need a new tool

Question

Does this always happen? Does φ have a carapoint at τ if and only if φ directionally differentiable at τ? Does a function ever have a linear approximation at a carapoint?

To answer these questions, we use a tool that allows us to avoid dealing with the function and instead analyze the geometry of vectors.

Hilbert spaces

A Hilbert space is an infinite dimensional analogue of a vector space. Hilbert spaces come equipped with inner products, orthogonality, and linear operators, the familiar tools of vector spaces.

In the early 1990s, J. Agler, following work of D. Sarason, invented the notion of a **Hilbert space model**, a tool for transforming questions about an analytic function into questions about inner products of vectors.

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such that the model equation

$$1 - \overline{\varphi(\mu)}\varphi(\lambda) = \left\langle \left(1 - \mu^* \lambda\right) u(\lambda), u(\mu) \right\rangle_{\mathcal{M}}$$

holds for all $\lambda, \mu \in \mathbb{D}^2$ *where* λ *is an operator from* $\mathbb{D}^2$ *into* $\mathcal{L}(\mathcal{M})$ *by*

$$\lambda = \lambda^1 P + \lambda^2 (1 - P),$$

where P *is a projection operator onto the space* $\mathcal{M}_1$.

J-C Theorem on the bidisk

Hilbert space models can be used to give a two-variable J-C Theorem:

Theorem (Agler, McCarthy, Young, 2012)

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   - c *for any model* $(\mathcal{M}, u)$ *of* φ*, the map* $u(\lambda)$ *is bounded on all sequences* $\lambda \overset{\text{nt}}{\to} \tau$*.*
2. φ *is nontangentially differentiable at* τ *if and only if for every model* $(\mathcal{M}, u)$ *for* φ*, the map* $u(\lambda)$ *extends continuously to* τ *on nontangential sets.*

What does it mean?

The two variable Julia-Carathéodory Theorem shows that a function can be linearly approximated at a carapoint precisely when Hilbert space models are continuous at that point (in the sense that $\lim_{\lambda \stackrel{\mathrm{nt}}{\to} \tau} u(\lambda) = u(\tau)$).

How do boundary singularities points play a role in this?

Singularities and differentiability

Example

$$\varphi(\lambda) = \frac{\lambda^1 + \lambda^2 - 2\lambda^1\lambda^2}{2 - \lambda^1 - \lambda^2}$$

has a singularity at $\chi = (1, 1)$ and is NOT nontangentially differentiable.

We might conjecture that this is always the case.

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Example

$$f(\lambda) = \frac{-4\lambda^1(\lambda^2)^2 + (\lambda^2)^2 + 3\lambda^1\lambda^2 - \lambda^1 + \lambda^2}{(\lambda^2)^2 - \lambda^1\lambda^2 - \lambda^1 - 3\lambda^2 + 4}$$

has a singularity at $\chi = (1,1)$ and IS nontangentially differentiable.

A more general model

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The model operator $\lambda = \lambda_1 P + \lambda_2(1 - P)$ is linear and does not share the singularity.

As a consequence, the Hilbert space $\mathcal{M}$ encodes the noise from the singular behavior instead of the operator λ, and the resulting space is too large.

Modeling singular carapoints

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Consider the function $I_Y(\lambda) : \mathbb{C}^2 \to \mathcal{L}(M)$ defined by

$$I_Y(\lambda) = \frac{\lambda^1 Y + \lambda^2(1 - Y) - \lambda^1\lambda^2}{1 - \lambda^1(1 - Y) - \lambda^2 Y}.$$

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- I_Y has a singular carapoint at χ.
- $I_Y(\chi) = I_{\mathcal{M}}$.

Use I_Y in a model

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That is, replace $(1 - \mu^* \lambda)$ with $(1 - I_Y(\mu)^* I_Y(\lambda))$ in the model.

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holds for all $\lambda, \mu \in \mathbb{D}^2$ where $I_Y(\lambda)$ is the operator valued map

$$I_Y(\lambda) = \frac{\lambda^1 Y + \lambda^2(1 - Y) - \lambda^1\lambda^2}{1 - \lambda^1(1 - Y) - \lambda^2 Y},$$

where Y is a positive contraction on $\mathcal{M}$.

Generalized models are continuous at carapoints

Theorem (T.D., '16 and Agler, T.D., Young, '12)

Let $\varphi \in \mathcal{S}_2$. $\chi = (1,1)$ is a carapoint for $\varphi \in \mathcal{S}_2$ if and only if there exists a generalized model $(\mathcal{M}, u, I)$ of φ such that u extends continuously to χ on nontangential sets.

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That is, we can always find a model function $u(\lambda)$ in a generalized model that extends continuously to the boundary. In other words, it makes sense to write

$$\lim_{\lambda \overset{\text{nt}}{\to} \chi} u(\lambda) = u(\chi).$$

We can use u to probe the behavior of φ at a singular carapoint.

A model argument

We calculate the directional derivative of φ at a carapoint χ in the direction $-\delta$: see board.

Slope functions

A Pick function is an analytic function from the complex upper halfplane into itself.

Theorem (Agler, McCarthy, Young and Agler, T.D., Young)

Suppose that $\varphi \in \mathcal{S}_2$ has a carapoint at χ. Let $-\delta$ be any direction pointing into the bidisk at χ. Then there exists a function h so that h and $-zh$ are in the Pick class so that

$$D_{-\delta}\varphi(\chi) = -\varphi(\chi)\delta_2 h(\frac{\delta_2}{\delta_1})$$

Singular points and differentiability

We have yet to answer our last question: how can we tell the difference between singular carapoints that give rise to linear approximations and those that give rise merely to directional derivatives?

Theorem (T.D., 16)

Let $\varphi \in \mathcal{S}_2$ have a carapoint at χ. $u(\chi) \perp \ker Y(1 - Y)$ if and only if φ is nontangentially differentiable at χ.

Theme: model geometry $\Leftrightarrow$ function theory

Fin

Thank you.

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