Welcome
from the
Department of Mathematics
at the
University of New Haven

Volumes and their boundaries in $\mathbb{R}^n$

Joseph Kolibal

The University of New Haven

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Abstract

The relationship of integrals on the surface of a region in $\mathbb{R}^n$ to integrals over the volume of the region is a fundamental part of calculus. We examine this relationship from the fundamental theorem of calculus and integration by parts, to the theorems of vector calculus, taking note of some interesting aspects of the technique and the diversity of results which can be obtained using these relationships.

In particular, the recursive nature of integration by parts is one of the building blocks of modern mathematics. It is a feature of so many problems that it deserves consideration for inclusion into Paul Halmos' table of the mathematical elements.

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We continue in Part II with examples from Calculus.

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  Let f be an integrable function on $[a,b]$. For $x \in [a,b]$, let $F(x) = \int_a^x f(t)dt$. Then F is continuous on $[a,b]$, and $F'(x)$ exists and equals $f(x)$ at every x at which f is continuous.

  Let F be continuous on $[a,b]$ and differentiable except at finitely many points in $[a,b]$, and let f agree with $F'(x)$ where F' is defined. If f is integrable on $[a,b]$, then $\int_a^b f(t)dt = F(b) - F(a)$.

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- The point to emphasize here is that the area or content of f on $[a,b]$ can be obtained, under suitably mild conditions, using only the endpoints of the interval $[a,b]$, i.e., using only the boundary of $[a,b]$ and some suitably constructed anti-derivative F.

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There is a fundamental link between volumes and boundaries.

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We examine this topic in greater detail.

- Given $f : [a,b] \mapsto \mathbb{R}$ and $g : [a,b] \mapsto \mathbb{R}$, continuously differentiable, then from the product rule we have

$$(f(x)g(x))' = f(x)g'(x) + g(x)f'(x).$$

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- Integrating both sides over $[a,b]$ gives the definite integrals

$$\int_a^b f(x)g'(x)dx = f(x)g(x)|_a^b - \int_a^b f(x)'g(x)dx.$$

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  Indeed, this is exactly what is needed.

- We start with the formula for integration by parts, i.e.,

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- Now if we choose φ so that $\varphi(a) = \varphi(b) = 0$, then we have

$$\begin{aligned}\int_a^b f'(x)\varphi(x)dx &= \cancel{f(x)\varphi(x)|_a^b}^{\,0} - \int_a^b f(x)\varphi'(x)dx,\\ &= -\int_a^b f(x)\varphi'(x)dx, \qquad (1)\end{aligned}$$

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$$= -\int_a^b f(x)\varphi'(x)dx, \qquad (1)$$

- We have integrated by parts, but we have also shifted the derivative from f to φ. If we make sure that φ is always differentiable, we can define f' using (1).

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- Let $f(a,b) \mapsto \mathbb{R}$, and $g(a,b) \mapsto \mathbb{R}$ be

[3] Integrable on closed on and bounded domains, e.g., a closed interval.

[4] The support of f, denoted $\text{supp}\,(f) = \{x \mid f(x) \neq 0\}$, is closed if $\text{supp}\, f = \overline{\{x \in (a,b) | f(x) \neq 0\}}$, and the support is compact if this set is bounded in $\mathbb{R}$. We have that φ and it's derivatives vanish outside $[a,b]$ and we say $\varphi \in C_0^\infty(a,b)$.

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Alright, let's require it to work for any suitable φ.

- Let $f(a,b) \mapsto \mathbb{R}$, and $g(a,b) \mapsto \mathbb{R}$ be local integrable functions.[3] We say that g is the weak derivative of f if

$$\int_a^b f(x)\frac{d\varphi(x)}{dx}dx = -\int_a^b g(x)\phi(x)dx \tag{2}$$

for all smooth functions $\varphi : (a,b) \mapsto \mathbb{R}$ with compact support,[4]

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So how does it work?

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- Applying the definition of weak derivative, we must find a function g such that

$$\int_{-1}^{1} |x|\varphi'(x)dx = -\int_{-1}^{1} g(x)\varphi(x)dx, \forall \varphi(x) \in C_0^\infty(-1,1). \quad (3)$$

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We begin by attempting to integrate the left side of (3) by breaking it up into an integral over $[-1,0]$ and $[0,1]$,

$$\int_{-1}^{1} |x|\varphi'(x)dx = \left(\int_{-1}^{0} (-x)\varphi'(x)dx + \int_{0}^{1} x\varphi'(x)dx \right). \quad (4)$$

- To integrate the first term on the right side of (4), we use integration by parts

$$\int_{-1}^{0}(-x)\varphi'(x)dx = \left(\underbrace{-x\varphi(x)\Big|_{-1}^{0}}_{\text{since } \varphi(-1)=0 \text{ and } x|_{x=0}=0} \to 0 \quad - \int_{-1}^{0}(-1)\varphi(x)dx \right).$$

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- To integrate the second term on the right side of (4), we use integration by parts

$$\int_{0}^{1} x\varphi'(x)dx = \left(\underbrace{\cancel{x\varphi(x)\Big|_{0}^{1}}}_{\text{since } \varphi(1)=0 \text{ and } x|_{x=0}=0} \to 0 - \int_{0}^{1}(1)\varphi(x)dx \right).$$

- Putting it together by equating these results to the right side of (3) , we have

$$(LHS): \int_{-1}^{1} |x|\varphi'(x)dx = \int_{-1}^{0}(1)\varphi(x)dx + \int_{0}^{1}(1)(-1)\varphi(x)dx$$
$$= \underbrace{-}_{\text{the minus sign in the definition}} \int_{-1}^{1} g(x)\varphi(x)dx \quad :(RHS).$$

and so by definition, we must have $g(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$.

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What if f has a jump, e.g., $f(x) = H(x)$, i.e., the Heaviside function?

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- Writing D_x for the weak derivative, we can show that $D_x H(x) = \delta(x)$, where $\delta(x)$ is the functional defined by

$$\int_{\mathbb{R}} f(x)\delta(x)dx = f(0).$$

- We note that the weak derivative of f is the same as the classical derivative when f is differentiable on (a,b). Quite a few details need to be added to fully make everything mathematically rigorous, however the discussion demonstrates the utility of integration by parts.

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It works, and it is not just a tool for evaluating integrals.

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- These relate the integral of a vector quantity over a volume to the the integral of an associated vector quantity over a surface of the volume.
- For example, the Divergence Theorem has

$$\int_{\Omega} \nabla \cdot \boldsymbol{F} dV = \int_{\partial\Omega} \boldsymbol{F} \cdot \boldsymbol{n} dS,$$

where $\Omega \subset \mathbb{R}^3$ is a suitably defined domain with a piecewise smooth, oriented boundary, $\partial\Omega$.

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We develop integration by parts in $\mathbb{R}^3$.

- Consider functions $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$, where $\boldsymbol{x} = (x, y, z) \in \mathbb{R}^3$, and examine the partial derivatives,

$$\frac{\partial}{\partial x} f(\boldsymbol{x}) g_x(\boldsymbol{x}) = f_x(\boldsymbol{x}) g_x(\boldsymbol{x}) + f(\boldsymbol{x}) g_{xx}(\boldsymbol{x}), \tag{5}$$

$$\frac{\partial}{\partial y} f(\boldsymbol{x}) g_y(\boldsymbol{x}) = f_y(\boldsymbol{x}) g_y(\boldsymbol{x}) + f(\boldsymbol{x}) g_{yy}(\boldsymbol{x}), \tag{6}$$

$$\frac{\partial}{\partial z} f(\boldsymbol{x}) g_z(\boldsymbol{x}) = f_z(\boldsymbol{x}) g_z(\boldsymbol{x}) + f(\boldsymbol{x}) g_{zz}(\boldsymbol{x}), \tag{7}$$

- Adding the left side of (5)–(7) gives

$$\underbrace{\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot}_{div = \nabla \cdot} \underbrace{f(\boldsymbol{x}) \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) g(x)}_{f\, grad\,(g)} = \nabla \cdot (f(\boldsymbol{x}) \nabla g(\boldsymbol{x})).$$

- Adding up the right sides in (5)–(7) gives

$$(f_x(\boldsymbol{x})g_x(\boldsymbol{x}) + f(\boldsymbol{x})g_{xx}(\boldsymbol{x})) + (f_y(\boldsymbol{x})g_y(\boldsymbol{x}) + f(\boldsymbol{x})g_{yy}(\boldsymbol{x})) + (f_z(\boldsymbol{x})g_z(\boldsymbol{x}) + f(\boldsymbol{x})g_{zz}(\boldsymbol{x})) = \nabla f(\boldsymbol{x}) \bullet \nabla g(\boldsymbol{x}) + f(\boldsymbol{x})\Delta g(\boldsymbol{x}).$$

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- Consequently, equating the left and right sides gives

$$\nabla \cdot (f(\boldsymbol{x})\nabla g(\boldsymbol{x})) = \nabla f(\boldsymbol{x}) \cdot \nabla g(\boldsymbol{x}) + f(\boldsymbol{x})\Delta g(\boldsymbol{x}),$$

and integrating gives

$$\int_\Omega \underbrace{\nabla \cdot (f(\boldsymbol{x})\nabla g(\boldsymbol{x}))}_{\nabla \cdot \boldsymbol{F} \text{ where } \boldsymbol{F} = f\,\nabla g} dV = \int_\Omega \nabla f(\boldsymbol{x}) \cdot \nabla g(\boldsymbol{x}) dV + \int_\Omega f(\boldsymbol{x})\Delta g(\boldsymbol{x}) dV.$$

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- Applying the divergence theorem (on p.13) to the left side of gives

$$\int_{\partial\Omega}(f(\boldsymbol{x})\nabla g(\boldsymbol{x}))\bullet\boldsymbol{n}dS = \int_{\partial\Omega} f(\boldsymbol{x})\frac{\partial g(\boldsymbol{x})}{\partial n}dS,$$ since $\nabla u\bullet\boldsymbol{n} = \partial u/\partial n.$

- Putting it together gives

$$\int_{\partial\Omega} f(\boldsymbol{x}) \frac{\partial g(\boldsymbol{x})}{\partial n} dS = \int_{\Omega} \nabla f(\boldsymbol{x}) \cdot \nabla g(\boldsymbol{x}) dV + \int_{\Omega} f(\boldsymbol{x}) \Delta g(\boldsymbol{x}) dV.$$

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This is known as Green's first identity.

- Using $f = u$, $\nabla g = v$, and noting that $\Delta = \nabla \bullet \nabla$, we can rewrite Green's first identity as

$$\int_{\Omega} \underbrace{f(\boldsymbol{x}) \Delta g(\boldsymbol{x})}_{udv = f\nabla\bullet\nabla g} dV = \underbrace{\int_{\partial\Omega} f(\boldsymbol{x}) \frac{\partial g(\boldsymbol{x})}{\partial n} dS}_{\text{boundary term}} - \int_{\Omega} \nabla \underbrace{f(\boldsymbol{x}) \bullet \nabla g(\boldsymbol{x})}_{(du)v} dV.$$

providing an analog of integration by parts in $\mathbb{R}^3$.

An important result in PDEs.

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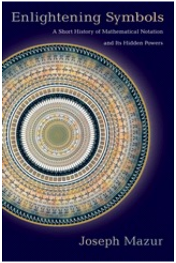
- Working with structures in mathematics requires acquiring the ability to manipulate ideas symbolically.
- Read, *Enlightening Symbols: A Short History of Mathematical Notation and Its Hidden Powers* by Joseph Mazur.



- If f is a continuous function on $[a,b]$ and F is any antiderivative of f on $[a,b]$, then

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- All of this is also reminiscent of state functions in thermodynamics, i.e., a property of a system that depends only on the current, equilibrium state of the system. For example, enthalpy and entropy are state functions. Thus, these are properties independent of path, i.e., they depend only on the starting and ending points of the thermodynamic system.

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Make your own art, find your own patterns.

Department of Mathematics at

The University of
New Haven

JKolibal@newhaven.edu
http://www.newhaven.edu

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