Simultaneous Triangularization of Collections of Operators and Matrices

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- A subspace $\mathcal{M}$ of a vector space $\mathcal{V}$ is said to be invariant under a linear operator $T : \mathcal{V} \to \mathcal{V}$ if $T\mathcal{M} \subseteq \mathcal{M}$

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Question. Dose every linear operator have a *nontrivial* invariant subspace?

Ans. No in general!

a. In finite dimensional spaces, it depends to the coefficient field. For $\mathcal{V} = \mathbb{R}^2$, as a vector space over $\mathbb{R}$, the rotation operator T_θ by an angle θ, $\theta \neq 0, \pi$, has no nontrivial invariant subspace.

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b. In infinite dimensions, there are linear operators with no nontrivial "closed" invariant subspaces.

- Per Enflo [1975] and C.J. Read [1984-8] showed that there are bounded linear operators on nonrefelexive Banch spaces with no nontrivial invariant subspaces.

- An operator T (or a collection $\mathcal{C}$ of operators) is said to be (simultaneously) triangularizable if there is a chain of invariant subspaces for T (for all members of $\mathcal{C}$) which is maximal as a subspace chain:

$$T = \begin{bmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ 0 & \vdots & \ddots & * \\ 0 & 0 & \cdots & * \end{bmatrix}$$

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- For such vector spaces, it is well known that any *commutative* collection $\mathcal{C}$ of operators or matrices is simultaneously triangularizable.

Question. Under what other conditions a collection $\mathcal{C}$ of matrices or operators is simultaneously triangularizable?

Observation 1. If $\mathcal{C}$ is commutative, then for all S and T in $\mathcal{C}$ the *commutator operator* $[S, T] = ST - TS = 0$.

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Observation 2. If $\mathcal{C}$ is triangularizable, then for all S and T in $\mathcal{C}$ the operator $[S, T] = ST - TS$ is nilpotent:

$$ST - TS = \begin{bmatrix} 0 & * & \cdots & * \\ 0 & 0 & \cdots & * \\ 0 & \vdots & \ddots & * \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

- In view of these observations, one way to generalize is to consider collections $\mathcal{C}$ for which the commutator $[S, T] = ST - TS$ is "small', in some sense'.

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- Such generalizations have been studied extensively and there are many interesting results. Here, we will survey some of them.

- In view of these observations, one way to generalize is to consider collections $\mathcal{C}$ for which the commutator $[S, T] = ST - TS$ is "small', in some sense'.
- Such generalizations have been studied extensively and there are many interesting results. Here, we will survey some of them.
- In the following parts I-IV,all operators are linear and defined on a complex f.d.v.s. $\mathcal{V}$. Also, $\mathcal{L}(\mathcal{V})$ stands fo the algebra of all such operators.

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Theorem(H.Radjavi 2000.) A semigroup $\mathcal{S}$ of operators on $\mathcal{V}$ is simultaneousely triangularizable iff each pair S, T of $\mathcal{S}$ is.

II. Algebras and Radicals

Definition. The *Radical* of a unital algebra $\mathcal{A}$, $Rad\mathcal{A}$, is

$$\{B \in \mathcal{A} : \sigma(AB) = \{0\} \textit{for all } A \in \mathcal{A}\}$$

$$= \{B \in \mathcal{A} : \sigma(BA) = \{0\} \textit{for all } A \in \mathcal{A}\} \ .$$

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Theorem(McCoy,1936.) If $\mathcal{A}$ be a unital subalgebra of $\mathcal{L}(\mathcal{V})$, then $\mathcal{A}$ is simultaneously triangularizable iff the quotient algebra $\mathcal{A}/Rad\mathcal{A}$ is commutative.

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Theorem(Laffey, 1979.) If $ST - TS$ has rank at most 1, then S, T are simultaneousely triangularizable.

Corollary A semisimple unital subagebra of $\mathcal{L}(\mathcal{V})$ is triangularizable iff it is commutative.

III. Rank ≤ 1 commutators

Theorem(Laffey, 1979.) If $ST - TS$ has rank at most 1, then S, T are simultaneousely triangularizable.

- In a just submitted paper (A. Jafarian, A. Popov, and H. Radjavi, 2016) we investigated existence of invariant subspaces as well as triangularizability under a similar rank condition:

Theorem(JPR, 2016). Suppose that $\mathcal{K}$ and $\mathcal{L}$ are two commutative sets of matrices and in addition $\mathcal{L}$ is a linear space. If $rank(AB - BA) \leq 1$ for all $A \in \mathcal{K}$ and $B \in \mathcal{L}$, then $\mathcal{K} \cup \mathcal{L}$ is simultaneousely triangularizable.

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Theorem.(JPR) Let $\mathcal{A}$ and $\mathcal{B}$ be algebras of matrices such that $rank(AB - BA) \leq 1$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. If $[A, B]$ is nonzero for some $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then $\mathcal{A} \cup \mathcal{B}$ has a common nontrivial invariant subspace.

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Theorem(Jafarian, Radjavi, Rosenthal, Sourour, 1995). Let $\mathcal{A}$ be an algebra of upper triangular matrices relative to a given basis. Then for every $\epsilon > 0$ there is an invertible matrix S_ϵ such that for all $A \in \mathcal{A}$

$$\|S_\epsilon^{-1} A S_\epsilon - \mathcal{D}(A)\| \leq \epsilon \|A\|,$$

where $\mathcal{D}(A)$ is the diagonal matrix of A.

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Theorem(JRRS, 1995). Suppose that $\mathcal{C}$ is a family of linear operators with the property that, for each finite subfamily $\{A_1, ..., A_m\}$, there is a constant $K > 0$ such that for every $\epsilon > 0$ there exist a commutative family $\{D_1, ..., D_m\}$ and an invertible S satisfying $\|S^{-1}A_jS - D_j\| < \epsilon$ and $\|D_j\| < K$ for all j. Then $\mathcal{C}$ is triangularizable.

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Theorem(E.Nordgreen, H.Radjavi, P.Rosenthal 1976.) Suppose S, T are compact operators on a Banach space. Then S, T is simultaneousely triangularizable iff $p(S, T)(ST - TS)$ is quasinilpotent for every noncommutative polynomial p.

Theorem(H. Radjavi, P. Rosenthal, V. Shulman 2000) A semigroup $\mathcal{S}$ of compact operators on a Banach space $\mathcal{X}$ is simultaneousely triangularizable iff for all S, T in $\mathcal{S}$, the commutator $ST - TS$ is quasinilpotent.

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Note The same conclusion hods for a semigroup of algebraic operators.

Theorem(A. Jafarian, H.Radjavi, P.Rosenthal, A.Sourour, 1995). Let $\mathcal{F}$ be a family of compact operators on a Banach space with the following property: Given any finite subset $\{A_1, ..., A_k\}$ of $\mathcal{F}$ there is a constant $M > 0$ such that for every $\epsilon > 0$ there are commuting compact operators $\{D_1, ..., D_k\}$ with $\|D_j\| \leq M$ and there is an invertible S satisfying

$$\|S^{-1}A_jS - D_j\| < \epsilon$$

for every j. Then $\mathcal{F}$ is triangularizable.

Theorem(A. Jafarian, H.Radjavi, P.Rosenthal, A.Sourour, 1995). If $\{A_1, ..., A_k\}$ is a simultaneousely triangularizable family of compact operators on a Hilbert space, then there exist commuting compact normal operators $\{D_1, ..., D_k\}$ such that for every $\epsilon > 0$ there is an invertible operator S satisfying

$$\|S^{-1}A_jS - D_j\| < \epsilon$$

for $j = 1, 2, ..., k$.