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Exploring Bounds for the Frobenius Number

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Example

Years ago, chicken Mcnuggets were sold in packs of $6, 9$, or 20. Could you have bought 30 McNuggets? 37 Mcnuggets? What is the largest number that you could not have bought?

Answers: $30 = 5 \times 6$, $37 \neq 6i + 9j + 20k$ for all $i, j, k \in \mathbb{N}_0$, largest number we could not have bought is 43.

The Frobenius Problem

- Let $G = \{a_1, a_2, \ldots, a_k\} \subset \mathbb{N}$ such that $\gcd(a_1, \ldots, a_k) = 1$;
- Find the largest integer that cannot be written as a non-negative linear combination of elements from G. We call this integer the Frobenius Number and we will denote it by $f(G)$.

Literature

- [Sylvester]: When $G = \{x, y\}$, $f(G) = xy - x - y$.
- When $G = \{x, y, z\}$, there is no explicit formula for $f(G)$ but there are lower and upper bounds in some cases:

  [Davison]: **Lower bound:** $\sqrt{3xyz} - x - y - z$.

  [Beck et al.]: **Conjectured Upper bound:** $(\sqrt{xyz})^{\frac{5}{4}} - x - y - z$.

- We used three different sequences to test these bounds:

  Arithmetic Sequence: $G = \{a, a+1, a+4\}$;

  Geometric Sequence: $G = \{a^2, ab, b^2\}$;

  Compound Sequence: $G = \{a_1a_2, b_1a_2, b_1b_2\}$;

- [Tripathi '17]: The Frobenius number for the arithmetic sequence:

$$f(a, a+1, a+4) = \begin{cases} \frac{1}{4}(a^2 + 8a - 4) & \text{if } a \equiv 0 \mod 4 \\ \frac{1}{4}(a^2 + 7a - 8) & \text{if } a \equiv 1 \mod 4 \\ \frac{1}{4}(a^2 + 6a - 12) & \text{if } a \equiv 2 \mod 4 \\ \frac{1}{4}(a^2 + 5a - 4) & \text{if } a \equiv 3 \mod 4 \end{cases}$$

- [Tripathi '08]: The Frobenius number for the geometric sequence:

$$f(a^2, ab, b^2) = a^2b + ab^2 - a^2 - b^2 - ab$$

Our Findings - Arithmetic Sequence

Arithmetic Sequence data encompassing 10,000 cases generated from GAP.

Frobenius Number and Upper & Lower Bounds Data for Arithmetic Sequence						
a	a+1	a+4	Frobenius Number	Lower Bound	Upper Bound	$xyz^{1/2}$
1	2	5	DNE	-2.52	-3.78	3.16
3	4	7	5	1.87	1.95	9.17
7	8	11	20	16.99	29.40	24.82
9	10	13	34	27.25	50.72	34.21
13	14	17	63	52.34	107.91	55.62
15	16	19	74	66.96	143.58	67.53
19	20	23	113	99.93	228.70	93.49

Figure 1: Sample Arithmetic Sequence Table

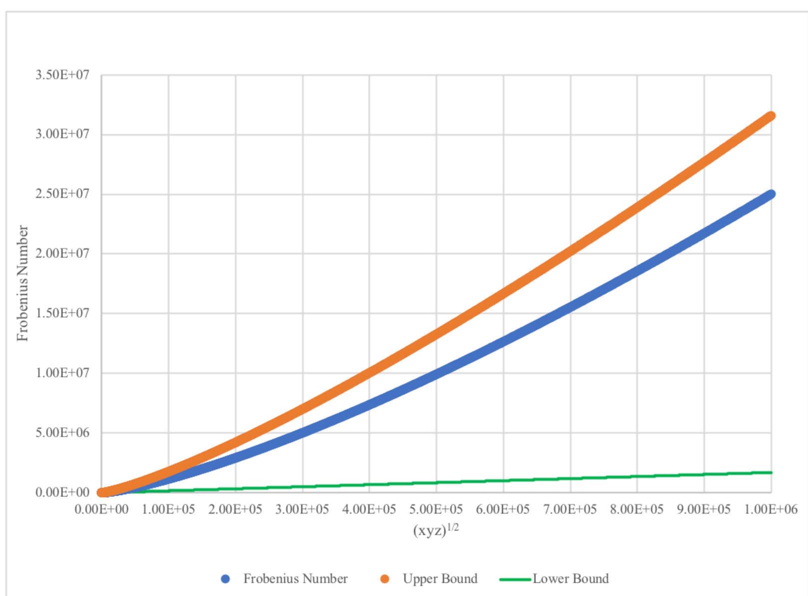

Figure 2: Arithmetic Sequence Graph

Our Findings - Geometric Sequence

Geometric Sequence data encompassing 10,000 cases generated from GAP.

Frobenius Number and Upper & Lower Bounds Data for Geometric Sequence						
a^2	ab	b^2	Frobenius Number	Lower Bound	Upper Bound	$xyz^{1/2}$
4	6	9	11	6.46	9.78	14.70
4	10	25	31	15.77	35.99	31.62
4	14	49	59	23.73	73.93	52.38
4	18	81	95	29.27	122.75	76.37
4	22	121	139	31.73	181.88	103.19
4	26	169	191	30.63	250.86	132.57
4	30	225	251	25.60	329.30	164.32

Figure 3: Sample Geometric Sequence Table

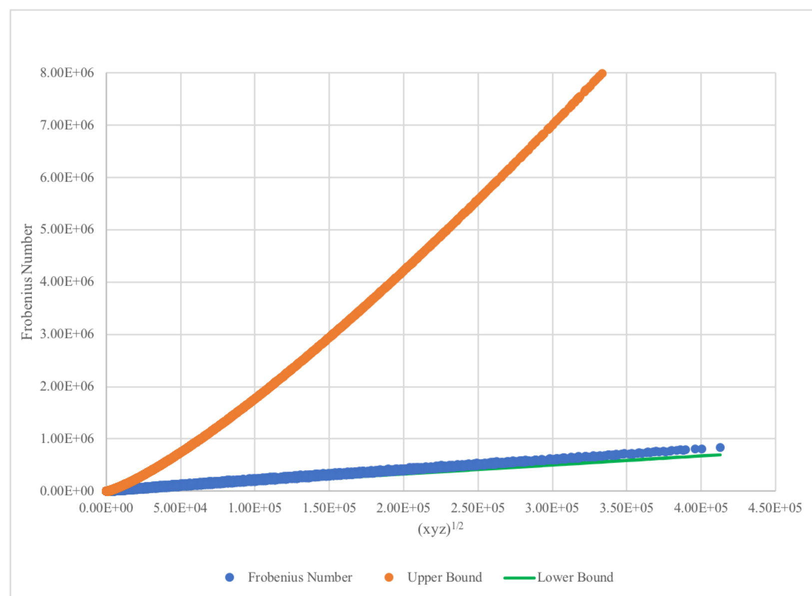

Figure 4: Geometric Sequence Graph

Our Findings - Compound Sequence

Compound Sequence data encompassing 10,000 cases generated from GAP.

Frobenius Number and Upper & Lower Bounds Data for Compound Sequence						
a_1a_2	a_1b_2	b_1b_2	Frobenius Number	Lower Bound	Upper Bound	$xyz^{1/2}$
4	6	9	11	6.46	9.78	14.70
4	6	15	17	7.86	14.60	18.97
4	6	21	23	7.88	17.87	22.45
4	6	27	29	7.09	20.18	25.46
4	10	15	21	13.43	25.49	24.49
4	10	25	31	15.77	35.99	31.62
4	10	35	41	15.81	43.54	37.42

Figure 5: Sample Compound Sequence Table

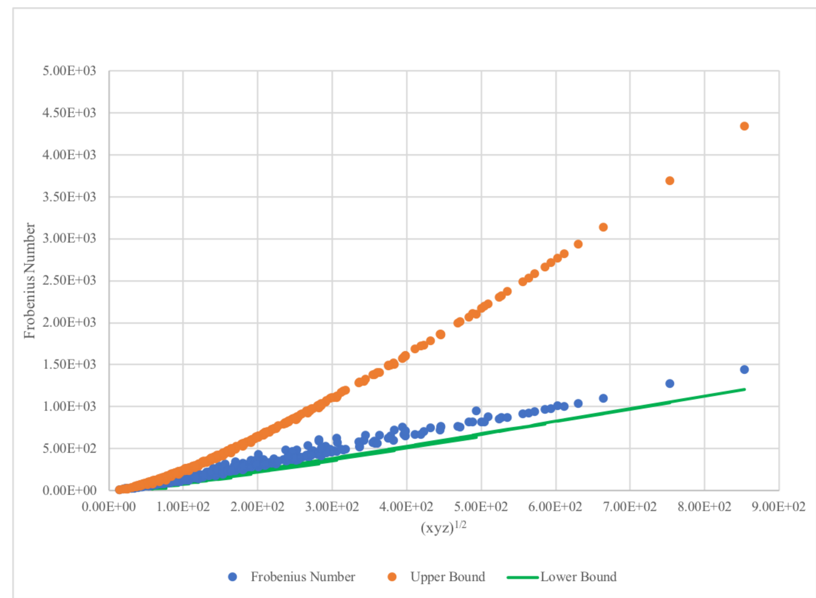

Figure 6: Compound Sequence Graph

Conclusion

- The upper bound for the Frobenius number of various sequences works in most cases. In some smaller cases, the upper bound was smaller than the Frobenius number.
- The upper bound for the Frobenius number seems to be a closer approximation than the lower bound for the arithmetic sequence.
- The lower bound for the Frobenius number seems to be a closer approximation than the upper bound for geometric sequences and compound sequences.

References

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5. A. Tripathi. On the Frobenius problem for geometric sequences. *Integers: Electronic J. of Combinatorial Number Theory*, 8:A43 (2008).